Part 2.1 Continuous functions

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Continuity

Definition 2.1.1 *A function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, is* ***continuous at*** $a \in A$ *if, and only if,*

$$\lim_{x \to a} f(x) = f(a) \,,$$

that is

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in A, |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon. \tag{1}$$

Notes

- Implicit in the definition is that $\lim_{x \to a} f(x)$ exists, i.e. is finite.
- In the definition of $\lim_{x \to a} f(x)$ we have the assumption that

  $$0 < |x - a| < \delta$$

  whereas in (1) we only have

  $$|x - a| < \delta.$$

  This is because in the definition of continuity we are assuming that $f(a)$ exists and, if $x = a$, then the requirement $|f(x) - f(a)| < \varepsilon$ is simply $0 < \varepsilon$ which is trivially true. Thus there is no need to exclude the possibility that $x = a$, and so we drop the requirement that $0 < |x - a|$.
- f is defined on some neighbourhood of a, *including* a,

Definition 2.1.2 *A function f is* ***continuous on an open interval*** (a, b) *if $f(x)$ is defined for, and is continuous at, every x in (a, b), i.e. for all c in (a, b) we have $\lim_{x \to c} f(x) = f(c)$.*

A function f is ***continuous on a closed and bounded interval*** $[a, b]$ *if $f(x)$ is defined for every x in $[a, b]$, and is continuous at every point in (a, b), i.e. for all c in (a, b) we have $\lim_{x \to c} f(x) = f(c)$, while $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.*

To prove a given function is continuous on an interval we need verify the definition of continuity for each point of the interval.

Results in Part 1 can be rephrased as

Example 2.1.3 *All polynomials are continuous on* $\mathbb{R}$.

All rational functions are continuous at points at which they are defined.

Solution is immediate. We saw in the last section that $\lim_{x\to a} p(x) = p(a)$ for all polynomials p at all points $a \in \mathbb{R}$. We also saw that $\lim_{x\to a} r(x) = r(a)$ all rational functions $r(a)$ at all points $a \in \mathbb{R}$ for which $r(a)$ is defined. ■

From an earlier result, f is continuous at a iff

$$\lim_{x\to a+} f(x) = f(a) = \lim_{x\to a-} f(x)\,. \tag{2}$$

This is used in the example of a function which is constructed from polynomials.

Example 2.1.4 *The function*

$$f(x) = \begin{cases} x^2 - x - 1 & \text{for } x \le 1 \\ x^3 - 2 & \text{for } x > 1, \end{cases}$$

is continuous on $\mathbb{R}$.

Solution If $a > 1$ then, in some neighbourhood of a we have $f(x) = x^3 - 2$. This is a polynomial and so f is continuous at such a.

If $a < 1$ then, in some neighbourhood of a we have $f(x) = x^2 - x - 1$. This is a polynomial and so f is continuous at such a.

If $a = 1$ then there is no neighbourhood of 1 in which f is a polynomial. Instead we have to use (2). For the left hand limit,

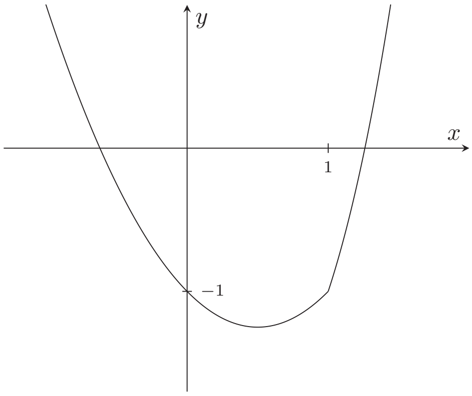
$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \left(x^2 - x - 1\right) = 1^2 - 1 - 1 = -1.$$

For the right hand limit

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \left(x^3 - 2\right) = 1^3 - 2 = -1.$$

And since these one-sided limits both equal $f(1) = -1$ we deduce that f is continuous at $x = 1$. ■

Graphically:



As a particular example of a rational function we have $(x^2-1)/(x^2-x)$, which by Example 2.1.3 is continuous for all $x \neq 0, 1$. Can we assign values at $x = 0$ and 1 to make it continuous on **all** of $\mathbb{R}$?

Example 2.1.5 *Prove that the function* $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} \dfrac{x^2-1}{x^2-x}, & \text{if } x \neq 0, 1, \\ 2 & \text{if } x = 1, \end{cases}$$

is continuous at $x = 1$.

Solution in Tutorial. ■

Example 2.1.6 *If* $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ *is given by*

$$f(x) = \frac{x^2-1}{x^2-x},$$

there is no value for $f(0)$ *that will make* $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ *continuous at* $x = 0$.

Solution in Tutorial

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \left(1 + \frac{1}{x}\right) = +\infty.$$

Thus $\lim_{x \to 0+} f(x)$ does not exist and hence $\lim_{x \to 0} f(x)$ does not exist. Therefore f cannot be continuous at $x = 0$. ■

Example 2.1.7 *Prove that the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) = \begin{cases} \sin\left(\dfrac{\pi}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is ***not*** *continuous at* $x = 0$.

Solution in Tutorial Since $\lim_{x\to 0} g(x)$ does not exist there is no value of $g(0)$ that it can equal. ■

Nowhere Continuous

More interesting, perhaps, are examples where continuity fails, and fails "frequently". A popular example of such a "pathological" function is the following, first described by Dirichlet in 1829. Recall that in any interval of real numbers we can find a rational number and we can find an irrational number. (See the appendix to the notes of Part 1)

Example 2.1.8 *Define* $f : \mathbb{R} \to \mathbb{R}$, *by*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f *is nowhere continuous.*

Solution Let $a \in \mathbb{R}$ be given Assume f is continuous at a.

Choose $\varepsilon = 1/2$ in the definition of continuity to find $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < 1/2$.

Two cases. First, $a \in \mathbb{Q}$. This implies $f(a) = 1$ while in any interval, such as $|x - a| < \delta$, i.e. $(a - \delta, a + \delta)$, we can find an *irrational* x_0 for which $f(x_0) = 0$. (For a proof of this see the appendix.) Thus

$$1/2 > |f(x_0) - f(a)| = |0 - 1| = 1.$$

Contradiction.

Second case left to Tutorial. ■

Rules of Continuity

Given functions continuous at $c \in \mathbb{R}$ we know, in particular, that the limits of the functions exist at c. This means that we can simply apply the limit rules of section 1.3. In this way we get

Theorem 2.1.9 *(i) Suppose that both f and g are defined on a neighbourhood of $c \in \mathbb{R}$ and are continuous at c. Then*

Sum Rule*: the sum of continuous functions is continuous,*

$$f + g \text{ is continuous at } c,$$

Product Rule*: the product of continuous functions is continuous,*

$$fg \text{ is continuous at } c,$$

Quotient Rule*: the quotient of continuous functions is continuous where defined,*

$$\frac{f}{g} \text{ is continuous at } c, \text{ provided } g(c) \neq 0.$$

(ii) Suppose that both f and g are continuous on an interval, then $f + g$, $f - g$, fg and f/g are continuous on the interval with the proviso that in the quotient, g is non-zero on the interval.

Proof in Tutorial As an example of the proof I prove only the Sum Rule, the others I leave to the students (or see the Appendix).

Since f and g are continuous at $a \in \mathbb{R}$ we have $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. The Rules for *limits* justify

$$\begin{aligned}
\lim_{x\to a} (f+g)(x) &= \lim_{x\to a} (f(x) + g(x)) \quad \text{from definition of } f+g, \\
&= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \quad \text{from Sum Rule for } \textit{limits}, \\
&= f(a) + g(a) = (f+g)(a).
\end{aligned}$$

■

In the next result we move the question of continuity at a point c to continuity at the origin.

Lemma 2.1.10 *$f(x)$ is continuous at $x = c$ if, and only if, $f(x+c)$ is continuous at $x = 0$.*

Proof Following the definitions we have $f(x)$ is continuous at $x = c$ iff $\lim_{x\to c} f(x) = f(c)$ iff

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon.$$

Writing $y = x - c$ this becomes

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y, |y - 0| < \delta \implies |f(c + y) - f(c)| < \varepsilon,$$

which is the definition of $f(c+y)$ being continuous at $y = 0$. Relabelling y as x gives the stated result. ∎

Trigonometric Functions

Recall that we have shown

$$\lim_{\theta\to 0} \sin\theta = 0 = \sin 0$$

and

$$\lim_{\theta\to 0} \cos\theta = 1 = \cos 0,$$

hence $\sin\theta$ and $\cos\theta$ are continuous at 0.

Assumption To continue we assume *addition formula for sine*, namely

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

This can be proved using diagrams; see

https://www.cut-the-knot.org/triangle/SinCosFormula.shtml#PWW

for a couple of such proofs.

Example 2.1.11 $\sin\theta$ *is continuous on* $\mathbb{R}$.

Solution Let $a \in \mathbb{R}$ be given. By Lemma 2.1.10 to show $\sin x$ is continuous at a it suffices to show that $\sin(a+x)$ is continuous at $x=0$. To do this consider

$$\begin{aligned}\lim_{x\to 0}\sin(a+x) &= \lim_{x\to 0}(\sin a\cos x + \cos a\sin x)\\ &= \sin a\lim_{x\to 0}\cos x + \cos a\lim_{x\to 0}\sin x\\ &\qquad\qquad \text{by the Sum Rule for limits,}\\ &= \sin a\times 1 + \cos a\times 0\\ &= \sin a.\end{aligned}$$

Thus $\sin(a+x)$ is continuous at $x=0$ and hence $\sin x$ is continuous at a. Yet a was arbitrary so $\sin x$ is continuous on $\mathbb{R}$. ■

That $\cos\theta$ is continuous on $\mathbb{R}$ is the subject of a question on the Problem Sheet.

Exponential Function

Recall that we have shown $\lim_{x\to 0} e^x = 1 = e^0$ and so e^x is continuous at $x=0$.

Assumption To continue we assume that

$$\forall \alpha,\beta\in\mathbb{R},\ e^{\alpha+\beta} = e^\alpha e^\beta.$$

Given our definition of e^x as an infinite series this requires taking the product of two infinite series and rearranging the product as a series of the same form as the originals. The product we take is the so-called *Cauchy Product of Series*, and the result we need use is that "the limit of the Cauchy product of two absolutely convergent series is equal to the product of the limits of those series". It is a gap in your education that there is no time to prove this in earlier analysis courses and now has to be assumed.

Aside The requirement that the series be absolutely convergent is used to deduce that their product is absolutely convergent. A result you may have seen in earlier analysis courses is that "an absolutely convergent series can be rearranged". That this is a non-trivial result can be seen from the interesting result that a conditionally convergent series (convergent but not absolutely convergent) can be rearranged to converge to any given number! **End of Aside**

Example 2.1.12 *e^x is continuous on $\mathbb{R}$.*

Solution Let $a \in \mathbb{R}$ be given. By Lemma 2.1.10 to show e^x is continuous at a it suffices to show that e^{a+x} is continuous at $x = 0$. Yet

$$\lim_{x \to 0} e^{x+a} = \lim_{x \to 0} e^x e^a = e^a.$$

■

Composition of Functions

We now have a large collection of continuous functions; polynomials: rational functions: trig functions and the exponential. We can now increase this collection with the following

Theorem 2.1.13 ***Composite Rule for functions.*** *Assume that g is defined on a deleted neighbourhood of $a \in \mathbb{R}$ and $\lim_{x \to a} g(x) = L$ exists. Assume that f is defined on a neighbourhood of L and is continuous there. Then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right). \tag{3}$$

Proof Let $\varepsilon > 0$ be given. Since f is continuous at L there exists $\delta_1 > 0$ such that,

$$|y - L| < \delta_1 \implies |f(y) - f(L)| < \varepsilon. \tag{4}$$

The definition of $\lim_{x \to a} g(x) = L$ means that

"$\forall \varepsilon > 0, \exists\, \delta > 0$ such that something holds".

TRICK Choose $\varepsilon = \delta_1$ in the definition of $\lim_{x \to a} g(x) = L$ to find $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |g(x) - L| < \delta_1. \tag{5}$$

Combine (4) and (5) (using $g(x)$ in place of y in (4)) to get

$$\begin{aligned} 0 < |x - a| < \delta_2 \quad &\implies \quad |g(x) - L| < \delta_1 \\ &\implies \quad |f(g(x)) - f(L)| < \varepsilon. \end{aligned}$$

Thus we have verified the definition that

$$\lim_{x \to a} f(g(x)) = f(L) = f\left(\lim_{x \to a} g(x)\right).$$

■

Warning I often see attempted proofs where students try to first use the fact that $\lim_{x\to a} g(x)$ exists. This is doomed to failure. You **must** start with the fact that f is continuous at $\lim_{x\to a} g(x)$ to find a δ_1 which is then used, in place of ε, in the definition of $\lim_{x\to a} g(x)$. **Remember**, f first, then g second.

If we assume more, namely that g is *continuous* at a, then we can deduce more.

Theorem 2.1.14 ***Composite Rule for Continuous functions***. *Assume that g is defined on a neighbourhood of $a \in \mathbb{R}$ and is continuous there and assume that f is defined on a neighbourhood of $g(a)$ and is continuous there, then $f \circ g$ is continuous at a.*

Proof In the previous theorem we can now replace $\lim_{x\to a} g(x)$ by $g(a)$ wherever it is seen. In particular in the conclusion, so

$$\begin{aligned}
\lim_{x\to a} (f \circ g)(x) &= \lim_{x\to a} f(g(x)) \quad \text{by definition of convolution,} \\
&= f\Big(\lim_{x\to a} g(x)\Big) \quad \text{by the Theorem 2.1.13} \\
&= f(g(a)) \quad \text{since } g \text{ is continuous at } a, \\
&= (f \circ g)(a) \quad \text{by definition of convolution.}
\end{aligned}$$

■

Example 2.1.15 *Show that*

$$\sin\left(\frac{\pi}{x}\right)$$

is continuous on $\mathbb{R} \setminus \{0\}$.

Solution

$$\sin\left(\frac{\pi}{x}\right) = (f \circ g)(x) \quad \text{with} \quad g(x) = \frac{\pi}{x} \quad \text{and} \quad f(x) = \sin x.$$

By the Quotient Rule, $g(x)$ is continuous for all $x \neq 0$, while by the example above $f(x)$ is continuous for all x. Hence by the Composite Rule for Continuous functions we deduce the stated result. ■